Concepts / Mathematical Theorems For Practical Use In Pursuit Of The Profession Of Engineer

1. Present Taylor's formula for functions of one variable and how can be used in approximating functions by polynomials.

Answer:

Let $f : I \subset \mathbf{R} \to \mathbf{R}$, and $x_0 \in I$, where $f \in C^{n+1}(I)$. Then

$f(x) = T_n(x) + R_n(x)$ (*Taylor's formula*),

where T_n is the Taylor's polynomial of n[th] order, and R_n is the reminder:

$$T_n(x) = f(x_0) + \frac{x - x_0}{1!} f'(x_0) + \ldots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0),$$

$$R_n(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_0 + \theta(x - x_0)),\ 0 < \theta < 1.$$

It follows the approximation formula for $f(x)$ in a neighborhood V of x_0:

$$f(x) \cong T_n(x),$$

with the error $\varepsilon_n = \sup_{x \in V} |R_n(x)|$.

2. Define the notions of eigenvalue (or proper value) and eigenvector (or proper vector) on a linear operator.

Answer:

We consider the vector space V defined over the field **K** and the linear operator f : V $\to$ V. A vector $v \in$ V (different from the null vector of V) is called an eigenvector (or proper vector) of the operator f if there exists a scalar λ from **K** such that $f(v) = \lambda v$. The scalar λ is called an eigenvalue (or proper value) of f.

3. Specify how the extremes of a function of class C^2 of two variables can be found.

Answer:

The extremes of the function $u = u(x, y)$ are among the *stationary points*, namely the solutions of the system $\begin{cases} \dfrac{\partial u}{\partial x} = 0 \\ \dfrac{\partial u}{\partial y} = 0 \end{cases}$.

A stationary point is a point of *minimum* if in this point

$$\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 > 0 \text{ and } \frac{\partial^2 u}{\partial x^2} > 0,$$

and is a point of *maximum* if in this point

$$\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 > 0 \text{ and } \frac{\partial^2 u}{\partial x^2} < 0.$$

4. Define the following notions: arithmetical mean, weighted arithmetical mean and geometrical mean.

***Answer*:**

Let $\{x_1, x_2, \ldots, x_n\}$ be a non-empty set of records (real numbers) with non-negative wedges $\{p_1, p_2, \ldots, p_n\}$.

Weighted mean: $M_p = \dfrac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{p_1 + p_2 + \cdots p_n}$ (the elements with a greater weight have more contribution to the mean). We can simplify the above formula taking normalized weights $\sum_{i=1}^{n} p_i = 1$. In this case we have $M_p = \sum_{i=1}^{n} p_i x_i$

Arithmetical mean : M_a it is a particular case of the weight mean M_p when all weights are equals $p_n = \dfrac{1}{n}$.

We have $M_a = \dfrac{1}{n} \sum_{i=1}^{n} x_i = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$ (M_a indicates the central trend of a set numbers).

Geometrical mean : $M_g = \sqrt[n]{x_1, x_2, \cdots x_n}$ if $x_i > 0$, $i = \overline{1,n}$. The geometrical mean has the following geometric explanation: the geometrical mean $M_g = \sqrt{ab}$ of two numbers *a*, *b* ∈ **R**₊ represents the length of a square with the same area as a rectangle with lengths *a* and *b*.

5. Define the notion of the conditional probability, write and explain the Bayes's formula.

Answer:

Let $\{E, K, P\}$ a probability space and $A, B \in K$ two events with $P(A) \neq 0$. We call the probability of the event *B* conditioned by the event *A,* the expression:

$$P_A(B) = P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Let $S = \{B_1, B_2 \ldots B_n\}$ an events complete system. Therefore,

$E = \bigcup_{i=1}^{n} B_i, B_i \in K, B_i \bigcap B_j = \phi, i \neq j$. We say that the system *S* is a partition of the sure event *E*, and the events B_i are called outcomes.

Bayes's formula:

$$P_A(B_i) = \frac{P(B_i) \cdot P_{B_i}(A)}{\sum_{j=1}^{n} P(B_j) \cdot P_{B_j}(A)}$$

This formula returns the probability of an outcome in the hypothesis that the event *A* has occured, or, more precisely, the probability that to occur the event *A* to be conditioned by the outcome B_i .

6. Define for a discrete (and finite) random variable the following numerical characteristics: mean value, variance and standard deviation.

Answer:

Let ξ be a discrete (and finite) random variable with its probability distribution

$$\xi : \begin{pmatrix} x_1, x_2, \ldots, x_n \\ p_1, p_2, \ldots, p_n \end{pmatrix}, \sum_{i=1}^{n} p_i = 1, \;\; p_i = P(\xi = x_i)$$

Mean value: $M(\xi)=\sum_{i=1}^{n} x_i\, p_i$. The mean value represents a numerical value around which it's find a group of the values for this random variable.

Variance: $D^2(\xi)=\sigma^2=M\left[(\xi-M(\xi))^2\right]$.

Standard deviation: $D(\xi)=\sigma=\sqrt{D^2(\xi)}$.

The variance and the standard deviation are indicators which explain the "scattering" of the values for a random variable, giving information on the concentration degree of the values around to its mean value.

7. Define the Laplace transform and write the formula for the derivative.

***Answer*:**

If *f* is an original function, then its Laplace transform is

$$(Lf)\,(s)=\int_0^{\infty} f(t)e^{-st}\,dt$$

Image of the derivative: $(Lf^{'})\,(s)=s\,(Lf)(s)-f\,(0_+)$.

8. Define the Z transform (the discrete Laplace transform) and calculate its image for the unit-step signal.

***Answer*:**

If $\{f_n\}$ is an original sequence, then its *Z* transform is:

$$Z(f_n)\,(z)=\sum_{n=1}^{\infty} f_n z^{-n}.$$

For the unit-step signal

$$\sigma_n=\begin{cases}0, & n<0,\\ 1, & \mathrm{n}\ge 0, \qquad \mathrm{n}\in\mathbb{Z}\end{cases}$$

its *Z* transform is $Z(\sigma_n)\,(z)=\sum_{n=1}^{\infty} z^{-n}=\dfrac{1}{1-\dfrac{1}{z}}=\dfrac{z}{z-1},\quad \text{for } \;|z|<1.$

9. Polar, cylindrical and spherical coordinate systems.

Answer:

The conversion between the Cartesian coordinates (x, y) of a point in the plane and the polar coordinates (ρ, ϕ) of the same point is given by the relations :

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \end{cases},$$

where $\rho \in [0, \infty)$, $\phi \in [0, 2\pi)$.

The conversion between the Cartesian coordinates (x, y, z) of a point in three-dimensional space and the cylindrical coordinates (ρ, ϕ, z) of the same point is given by the relations :

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases},$$

where $\rho \in [0, \infty)$, $\phi \in [0, 2\pi)$, $z \in$ **R**.

The conversion between the Cartesian coordinates (x, y, z) of a point in three-dimensional space and the spherical coordinates (ρ, ϕ, θ) of the same point is given by the relations :

$$\begin{cases} x = \rho\cos\phi\sin\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\theta \end{cases},$$

where $\rho \in [0, \infty)$, $\phi \in [0, 2\pi)$, $\theta \in [0, \pi]$.

10. Physical and geometrical magnitudes calculated by integrals. Formula for the flux of a vector field.

Answer:

Area of a plane domain, volume of a body, mass, centre of gravity, moments of inertia, the work of a field of force.

Let S be a smooth surface and let $\vec{v} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a continuous vector field on S. The flux of the vector field $\vec{v}$ across the surface S oriented by the normal vector $\vec{n} = (\cos\alpha)\vec{i} + (\cos\beta)\vec{j} + (\cos\gamma)\vec{k}$ is:

$$\iint_S (\vec{v}\vec{n})dS = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS .$$

11. Derivative with respect to a versor of a real function. Gradient, divergence and curl.

Answer:

Let $f : D \subset \mathbf{R} \to \mathbf{R}$ be a scalar field, let $\vec{s} \in \mathbf{R}^3$, ||s||=1, be a versor and $\vec{a} \in D$. The derivative of *f* in the direction of $\vec{s}$ at the point $\vec{a}$ is the limit (provided that it exists)

$$\lim_{t\to 0} \frac{1}{t}[f(\vec{a}+t\vec{s}) - f(\vec{a})] := \frac{\partial f}{\partial \vec{s}}(\vec{a})$$

The derivative $\frac{\partial f}{\partial \vec{s}}(\vec{a})$ characterizes the velocity variation of f with respect to $\vec{s}$ at the point $\vec{a}$. The gradient of f at $\vec{a}$ is defined by

$$gradf(\vec{a}) = \nabla f(\vec{a}) = \frac{\partial f}{\partial x}(\vec{a})\vec{i} + \frac{\partial f}{\partial y}(\vec{a})\vec{j} + \frac{\partial f}{\partial z}(\vec{a})\vec{k}$$

where Nabla is the operator of Hamilton

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} .$$

It can be proved that $\frac{\partial f}{\partial \vec{s}}(\vec{a}) = \vec{s} \cdot \nabla f(\vec{a})$, that is the directional derivative of f at $\vec{a}$ in the direction $\vec{s}$ is equal to the dot product between the gradient of f and $\vec{s}$.

From here it follows that the gradient direction of a scalar field is the direction of maximum value of that field, that is the field has the fastest variation.

Let $\vec{v} : U \to \mathbf{R}$ be a vector field defined on an open set $U \subset \mathbf{R}^3$, $\vec{v} = (P, Q, R)$. The divergence of the field $\vec{v}$ at a current point is the scalar (number)

$$div\vec{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

The curl of the field $\vec{v}$ at a current point is the vector

$$curl\vec{v} = \nabla f(\vec{a}) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} .$$

12. Write the Fourier series and the Fourier coefficients for a continuous periodic signal.

***Answer*:**

Let $f : \mathbf{R} \to \mathbf{R}$ be an integrable and periodic function having the period T and $\omega = \frac{2\pi}{T}$.

The Fourier coefficients are:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt, \quad n = 0,1,\ldots$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt, \quad n = 1,2,\ldots$$

The Fourier series associated to f is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega x) + b_n \sin(n\omega x).$$

13. Define the Fourier transform. The Fourier inverting formula.

***Answer*:**

The Fourier transform of an absolutely integrable function $f : \mathbf{R} \to \mathbf{C}$ is:

$$\hat{f}(\omega) = \int_R f(t) e^{-it\omega} dt.$$

The Fourier inverting formula is

$$f(t) = \frac{1}{2\pi}\int_R \hat{f}(\omega) e^{it\omega} d\omega.$$

14. Write the filtering formula and the Fourier transform for the unit impulse.

Answer:

The filtering formula is: $\delta(x - x_0) = \delta_{x_0}$, where δ is the Dirac's distribution.

The Fourier transform is $\hat{\delta} = 1$.

15. Solve the Cauchy-Problem

$$\begin{cases} x'(t) = a(t)x(t) \\ x(t_0) = x_0 \end{cases}$$

where a is a continuous function.

Answer:

The given equation can be rewritten as

$$\frac{x'(s)}{x(s)} = a(s).$$

Integrating between t_0 and t, we obtain

$$\ln x(t) - \ln x(t_0) = \int_{t_0}^{t} a(s)ds \Leftrightarrow \ln \frac{x(t)}{x(t_0)} = \int_{t_0}^{t} a(s)ds.$$

Thus, the sought-for solution is

$$x(t) = x_o e^{\int_{t_0}^{t} a(s)ds} \quad .$$